Journal of Statistical Physics, Vol. 34, Nos. 1/2, 1984

Mellin Transforms Associated with Julia Sets and Physical Applications

D. Bessis,[1] J. S. Geronimo,[1,2] and P. Moussa[1]

Received August 30, 1983

We introduce the Mellin transform of the balanced invariant measure associated to the Julia set generated by a rational transformation. We show that its analytic continuation is a meromorphic function, the poles of which are on a semi-infinite periodic lattice. This allows one to have an understanding of the behavior of the measure near a repulsive fixed point. Trace identities corresponding to the fact that the analytically continued Mellin transform vanishes at negative integers are derived for the polynomial case. The quadratic map is first analyzed in detail, and the analytic properties of the inverse of the Green's function are exhibited. Of interest is the appearance of a dense set of spikes at dyadic points when the Julia set is disconnected. These results are used to study the residues of the Mellin transform. A certain number of physically interesting consequences are derived for the spectral dimensionality of quantum mechanical systems, the excitation spectrum of which displays unusual oscillations. The appearance of complex critical indices for thermodynamical systems is also discussed in the conclusion.

KEY WORDS: Functional equations; rational transformations; Julia sets; fractal dimension; spectral dimension; oscillatory critical behavior.

1. INTRODUCTION

Given a rational transformation of degree N,

$$z' = T(z) \tag{1.1}$$

where $T(z)$ is a rational fraction, one associates a Julia set[1] J, which is the

[1] Service de Physique Théorique, CEN-SACLAY, F91191 Gif sur Yvette, Cedex, France.
[2] Permanent address: School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332. Supported in part by a N.A.T.O. Postdoctoral fellowship.

0022-4715/84/0100-0075$03.50/0

closure of all unstable fixed points of all the iterates of $T(z)$. On this set there exists a unique invariant balanced probability measure $\mu(E)$, which satisfies[2,3]

$$\mu(E) = \mu(T^{-1}E) = N\mu\big(T_i^{-1}(E)\big) \tag{1.2}$$

where E is any Borel measurable subset of J, and

$$T^{-1}E = \bigcup_{i=1}^{N} T_i^{-1}E \tag{1.3}$$

the $T_i^{-1}E$ being the N pre-images of E by T. For definiteness we shall suppose the degree of the numerator of $T(z)$ equal to N, and bigger than or equal to the degree D of the denominator.

1.1. Electrostatic Description

In the polynomial case, one introduces an electrostatic language, in which the previous measure appears to be the equilibrium or maximum entropy measure, for a two-dimensional electrical system.[2,4] The corresponding Green's function is defined by[3]:

$$G_{\text{E.S.}}(z) = \int_J \ln(z - x)\, d\mu(x) \tag{1.4}$$

where E.S. stands for electrostatic. $G_{\text{E.S.}}$ is the solution of the functional equation:

$$G_{\text{E.S.}}(z) = \frac{1}{N} G_{\text{E.S.}}(T(z)) \tag{1.5}$$

which behaves as

$$G_{\text{E.S.}}(z) \sim \ln z \qquad \text{for} \quad z \to \infty \tag{1.6}$$

Besides $G_{\text{E.S.}}(z)$, the real part of which represents the potential in the electrostatic analogy, one can introduce the Böttcher function:[1]

$$B(z) = \exp(G_{\text{E.S.}}(z)) \tag{1.7}$$

and also[5] the generating function of the moments of $d\mu$:

$$g(z) = \int \frac{d\mu(x)}{1 - zx} \tag{1.8}$$

which is simply related to $G(z)$ by

$$g(z) = \frac{1}{z} G'\left(\frac{1}{z}\right) \tag{1.9}$$

[3] The Green's function is usually defined as the real part of (1.4). For convenience we use its complex extension.

The function $g(z)$ has the advantage of being uniform near the point at infinity.

As will be seen in Section 4, it is also useful to introduce the Mellin transform of $d\mu$ with respect to a point $z \in J$ by:

$$M_z(s) = \int_J (z - x)^s \, d\mu(x) \tag{1.10}$$

This function will allow a precise analysis of the behavior of the Green's function near a point of the Julia set. More precise and complete indications will be found in Section 2.

Besides the well-known electrostatic language, we want to emphasize two other possible languages for the previous functions—one is quantum mechanical, the other is thermodynamical.

1.2. Quantum Mechanical Description

We still restrict ourselves to the case where $T(z)$ is a polynomial of degree N in z. It is possible[5,6] to associate to $T(z)$ a one-dimensional quantum mechanical system on a lattice, with a Hamiltonian whose spectrum is invariant under T, and such that there exists a decimation operator D acting on the wave function, with the properties

$$DT(H) = HD \tag{1.11}$$

We, of course, suppose here that the Julia set is real, so that H can be Hermitian. In this case, the measure μ can be identified as the integrated density of states. We will introduce the following function associated to this problem:

$$G_{\text{Q.M.}}(z) = \left\langle \delta_0 \left| \frac{1}{1 - zH} \right| \delta_0 \right\rangle = \int_J \frac{d\mu(x)}{1 - xz} \tag{1.12}$$

Q.M. stands here for quantum mechanical, and $|\delta_0\rangle$ is the eigenfunction of a state localized at point 0.[6] We notice here a first possible confusion due to the fact that $G_{\text{Q.M.}}(z)$ is called the Green's function of the quantum mechanical system, although it has to be identified not with $G_{\text{E.S.}}(z)$ but with its derivative $g(z)$. We shall encounter similar misleading ambiguities with other functions, and a little dictionary (Table I) given at the end of this introduction will display the correspondence.

The "Lyapounov function"[7] defined as[4]

$$L(z) = \int_J \ln(z - x) \, d\mu(x) \tag{1.13}$$

[4] The "Lyapounov function" is the real part of (1.13).

is nothing but the previous $G_{\mathrm{E.S.}}(z)$ Green's function. The Fredholm determinant[8]

$$\Delta(z) = e^{L(z)} \tag{1.14}$$

was called in the electrostatic language the Böttcher function. The ζ function,[8] previously called Mellin transform,

$$\zeta(z) = \int_J (a - x)^s \, d\mu(x) \tag{1.15}$$

(where a is the end point of the spectrum) will give, by analysis of its singularities, an insight into the excitation spectrum (spectral dimensionality[9,10]) of the quantum mechanical system.

1.3. Thermodynamical Description

The Ising hierarchical lattices[11] lead to recursion relations for the partition function which involve the invariance of the thermodynamical functions under rational transformations. The Julia set associated with these transformations appears in the thermodynamical limit as the set of singularities of the free energy[12]:

$$F(y) = \int_J \ln(y - x) \, d\mu(x) \tag{1.16}$$

which corresponds to the electrostatic Green's function and to the "Lyapounov function" in the previous languages. Here (1.16) is a representation of the Lee–Yang[13] type, in which $d\mu(x)$ is the density of zeros of the partition function in the thermodynamic limit. Writing

$$T(y) = N(y)/D(y) \tag{1.17}$$

where the highest degree coefficient of $N(y)$ has been chosen equal to 1, we get

$$F(T(y)) = \int_J \ln(T(y) - x) \, d\mu(x) \tag{1.18}$$

Considering here the case where the degree N of $N(y)$ is greater than the degree D of $D(y)$, we can write

$$N(y) - xD(y) = \prod_{i=1}^{i=N} \left[y - T_i^{-1}(x) \right] \tag{1.19}$$

and therefore

$$\frac{1}{N} F(T(y)) = \frac{1}{N} \sum_{i=1}^{N} \int \ln\left(y - T_i^{-1}(x)\right) d\mu(x) - \frac{1}{N} \ln D(y) \tag{1.20}$$

However, the invariance tells us that

$$\frac{1}{N}\sum_{i=1}^{N}\int_J \ln\left(y - T_i^{-1}(x)\right) d\mu(x) = F(y) \tag{1.21}$$

and therefore we get

$$F(y) = \frac{1}{N}F(T(y)) + \frac{1}{N}\ln D(y) \tag{1.22}$$

Equation (1.22) reduces to the ordinary functional equation for the Green's function deduced in the polynomial case from the Böttcher equation. However, Eq. (1.22) in the general case is of a different nature, due to the presence of an inhomogeneous term, and also by the fact that the factor N is the degree of the numerator alone, and not the difference between the degrees of the numerator and denominator, which would occur in a Böttcher equation. Equation (1.22) involves the full Julia set and not only the outer part which limits the basin of attraction of the point at infinity.

The repulsive real fixed point of $T(y)$ will attract attention because it corresponds to the critical temperature.[12] The behavior of the solution $F(y)$ of (1.22) near this point will be a singular one which requires some mathematical analysis. Again $F(y)$ is fixed by its behavior $F(y) \sim \ln y$ near infinity. Following Derrida et al.,[14] one analyzes the solution (1.22) near the critical point as

$$G(y) \sim h\{\ln(y - y_c)\}(y - y_c)^{2-\alpha} \tag{1.23}$$

where h, the so-called amplitude, is here a periodic function of $\ln(y - y_c)$ and not merely a constant.

In this paper we will direct our attention primarily to the quadratic map:

$$T(y) = y^2 - \lambda \tag{1.24}$$

although many of the ideas and technicalities involved will also be applied to rational maps. We shall introduce in a systematic way the Mellin transform of the invariant measure:

$$M(s) = \int_J (y - y_c)^s \, d\mu(y) \tag{1.25}$$

from which all relevant properties near y_c of the free energy can be worked out.

Before giving a brief review of the content of this paper, we summarize the various notations in Table I. The reader is requested to notice that the ζ function introduced in the quantum mechanical language appears to be different from the ζ functions[15] introduced in the theory of dynamical systems; the situation being even more confusing for the Green's functions.

Table I. Notations

Electrostatic Language	Quantum Mechanical Language	Thermodynamical Language
$\mu(E)$: electric charge supported by the set E	$\mu(E)$: integrated density of states	$\mu(E)$: integrated density of zeros of the partition function
$G_{\mathrm{ES}}(z) = \int_J \ln(z - x)\, d\mu$ = electrostatic Green's function[a]	$L(z) = \int_J \ln(z - x)\, d\mu$ = "Lyapounov" function[a]	$F(z) = \int_J \ln(z - x)\, d\mu$ = free energy per site
Böttcher function = exponential of the Green's function	Fredholm determinant = exponential of the "Lyapounov" function	Partition function per site = exponential of the free energy
$g(z) = \int_J \frac{d\mu}{1 - zx}$; generating function of the moments	$R(z) = \int_J \frac{d\mu}{1 - zx}$; resolvent, or quantum mechanical Green's function	$U(z) = \int_J \frac{d\mu}{1 - zx}$; internal energy
$M_z(s) = \int_J (z - x)^s\, d\mu$; Mellin transform	$\zeta_z(s) = \int_J (z - x)^s\, d\mu$; ζ function	$M_z(s) = \int (z - x)^s\, d\mu$; Mellin transform

[a] It is the real part of these functions which are in general considered.

In Section 2 we introduce precise definitions and notations. The functional properties of the inverse of the electrostatic Green's function ("Lyapounov function" in the quantum mechanical language) are derived.

In Section 3, the analyticity and positivity properties of the functional inverse of the Green's function are analyzed, for the quadratic map, both in the case $\lambda < 2$ where the Julia set is connected and for the case $\lambda > 2$ where it is a Fatou's dust. The transition point $\lambda = 2$ appears as a bifurcation, and a dense set of "spikes" with dyadic values appears.

In Section 4, we analyze the heuristic content of the general solution of the functional equation fulfilled by the electrostatic Green's function:

$$G(z) = \tfrac{1}{2} G(z^2 - \lambda) \tag{1.26}$$

The general solution of (1.26) depends on an arbitrary periodic function.

In Section 5, we introduce the Mellin transform, with respect to a repulsive fixed point, of the invariant measure $d\mu$. First we analyze in great detail the case of the quadratic map. We show that the Mellin transform extends into a meromorphic function, the poles of which are on a semi-infinite periodic rectangular lattice. We also analyze some positivity properties of the residues on the real negative axis.

In Section 6, we give a generalization of the meromorphy property of $M(s)$ for a large class of rational transformations.

In Section 7, we show that there exist "trace identities",[8] in the polynomial case, which express the fact that the analytic continuation of the Mellin transform vanishes at negative integers.

In the conclusion, we derive a certain number of physically interesting consequences, for the quadratic mapping Hamiltonian model,[6] and for the Serpinsky gasket mechanical model,[16] for which it appears that the spectral dimension is in a sense necessarily complex, while, for the hierarchical thermodynamical models, one expects complex critical indices.[17]

2. DEFINITIONS AND NOTATIONS

We shall, for definiteness, concentrate our attention on the quadratic transformation:

$$T(z) = z^2 - \lambda \tag{2.1}$$

for λ real, although many of our results extend thoroughly to more general cases. The Böttcher function,[1] is the solution of

$$B^2(z) = B(z^2 - \lambda) \tag{2.2}$$

which is analytic around z infinity, and satisfies for large z:

$$B(z) = z + \sum_{k=1}^{\infty} \beta_k / z^k \tag{2.3}$$

When the Julia set is connected, this function maps the exterior of the set on the exterior of the unit circle.[1] We shall see later on, in detail, how to modify this statement when the Julia set is not connected.

Introducing the Green's function, as defined in (1.4), by

$$G(z) = \ln B(z) \tag{2.4}$$

we have

$$G(z) = \ln z - \sum_{n=1}^{\infty} \frac{g_n}{n z^n} \tag{2.5}$$

which converges for $|z|$ sufficiently large. We have

$$G(z) = \tfrac{1}{2} G(z^2 - \lambda) \tag{2.6}$$

It is convenient to introduce, instead of the derivative $G'(z)$, the function $g(z)$:

$$g(z) = \frac{1}{z} G'\left(\frac{1}{z}\right) \tag{2.7}$$

the expansion of which, for z sufficiently small, reads

$$g(z) = 1 + \sum_{n=1}^{\infty} g_n z^n \tag{2.8}$$

$g(z)$ is the solution of the functional equation:

$$g(z) = \frac{1}{1 - \lambda z^2} g\left(\frac{z^2}{1 - \lambda z^2}\right), \qquad g(0) = 1 \tag{2.9}$$

which is analytic around $z = 0$.

For λ real bigger than 2, $g(z)$ has the following spectral representation:

$$g(z) = \int_{-a}^{+a} \frac{d\mu(x)}{1 - zx} \tag{2.10}$$

where $d\mu(x)$ is the balanced invariant equilibrium measure defined on the Julia set.(18) The abscissa a of the rightmost point of the Julia set in the complex plane is

$$a = a(\lambda) = \frac{1 + (1 + 4\lambda)^{1/2}}{2} \tag{2.11}$$

The point a is a fixed point of (2.1). In general, $g(z)$ is given by

$$g(z) = \int_J \frac{d\mu(x)}{1 - xz} \tag{2.12}$$

and $G(z)$ by

$$G(z) = \int_J \ln(z - x)\, d\mu(x) \tag{2.13}$$

However, in the case $\lambda > 2$, the interest of introducing the function $g(z)$ is due to the fact that, the support of the measure $d\mu$ being a Cantor set, $g(z)$ is a uniform function of z.(19) Its singularities are only essential ones, staying on the Julia set. On the contrary the function $G(z)$ is multisheeted, and therefore its structure is less transparent.

Besides these functions we want to introduce the functional inverse of $G(z)$:

$$E(z) = G^{(-1)}(z) \tag{2.14}$$

which fulfills

$$E(2z) = E^2(z) - \lambda \tag{2.15}$$

an equation of the Poincaré–Picard type.[20] From (2.5), we have to choose the solution of (2.5) which is analytic for $\operatorname{Re} z$ sufficiently large, and admits an expansion of the following form:

$$E(z) = \sum_{k=-1}^{+\infty} \gamma_k e^{-kz} \tag{2.16}$$

with

$$\gamma_{-1} = +1 \tag{2.17}$$

Furthermore, $E(z)$ satisfies the following relations:

$$E^*(z) = E(z^*) \tag{2.18}$$

$$E(z + i\pi) = -E(z) \tag{2.19}$$

Equation (2.19) means that $E(z)$ is periodic of period $2i\pi$, and that the γ_{2p} vanish in (2.16). In order to check these relations, we need first to introduce the functional inverse $\epsilon(B)$ of the Böttcher function, which from (2.2) satisfies

$$\epsilon(B^2) = \epsilon^2(B) - \lambda \tag{2.20}$$

Using (2.3), we find

$$\epsilon(B) = \sum_{k=-1}^{\infty} \gamma_k / B^k, \qquad \gamma_{2p} = 0, \qquad \gamma_{-1} = 1 \tag{2.21}$$

Equations (2.16), (2.18), (2.19) then follow from $E(z) = \epsilon(e^z)$.

3. POSITIVITY AND ANALYTICITY PROPERTIES OF THE INVERSE GREEN'S FUNCTION

We consider now the case where λ is a real number, which is bigger than $-1/4$. We shall see that a bifurcation occurs for $E(z)$ at $\lambda = 2$. We study the function $E(z)$ solution of

$$E(2z) = E^2(z) - \lambda \tag{3.1}$$

$$E^*(z^*) = E(z) \tag{3.2}$$

$$E(z + i\pi) = -E(z) \tag{3.3}$$

which for $\operatorname{Re} z$ sufficiently large, admits the convergent expansion:

$$E(z) = \sum_{k=-1}^{+\infty} \gamma_k e^{-kz}, \qquad \gamma_{-1} = +1 \tag{3.4}$$

3.1. Positivity Properties

When z increases from zero to $+\infty$ on the real axis $E(z)$ increases monotonically from a to $+\infty$ in a monotonic way. This is so because $G(z)$, which is the inverse function of $E(z)$, increases monotonically from 0 to $+\infty$ when z varies from a to $+\infty$. The monotonicity properties of $G(z)$ on the real axis follow from the fact that it is real on the real axis, and its real part being harmonic cannot have a maximum. An equivalent argument follows from the fact that $B = e^G$ is a conformal mapping. By (3.3), we deduce that $E(z)$ decreases monotonically from $-a$ to $-\infty$ when $z = i\pi + \xi$, and ξ increases from 0 to $+\infty$.

Since the function $E(z)$ is periodic with period $2i\pi$, and real symmetric, it is sufficient to study it in the strip $0 \leqslant \operatorname{Im} z \leqslant i\pi$. We notice that, from the expansion (3.4), $E(z)$ behaves like e^z near infinity; therefore given $\epsilon > 0$, there exists $M > 0$ such that for

$$\begin{gathered} \operatorname{Re} z > M, \qquad \epsilon < \operatorname{Im} z < \pi - \epsilon \\ \operatorname{Im} E(z) > 0 \end{gathered} \tag{3.5}$$

But from (3.1) we get

$$\operatorname{Im} E(z) = 2 \operatorname{Im} E(z/2) \operatorname{Re} E(z/2) \tag{3.6}$$

Equation (3.6) allows us to get rid of the ϵ in (3.5): If there exists η, $0 < \eta < \epsilon$ such that $\operatorname{Im} E(x + i\eta) = 0$ and $x > M$, then (3.6) tells us that $\operatorname{Im} E(2^p x + 2^p i\eta) = 0$, which contradicts (3.5) for p large enough. A similar argument holds for the real part of E, and we have:

$$\begin{gathered} \left.\begin{array}{l} \operatorname{Re} z > M \\ 0 < \operatorname{Im} z < \frac{\pi}{2} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \operatorname{Im} E(z) > 0 \\ \operatorname{Re} E(z) > 0 \end{array}\right. \\ \left.\begin{array}{l} \operatorname{Re} z > M \\ \frac{\pi}{2} < \operatorname{Im} z < \pi \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \operatorname{Im} E(z) > 0 \\ \operatorname{Re} E(z) < 0 \end{array}\right. \end{gathered} \tag{3.7}$$

Let us call $S(M; \alpha, \beta)$ the strip $M < \operatorname{Re} z$, $\alpha < \operatorname{Im} z < \beta$. We know that $\operatorname{Im} E$ and $\operatorname{Re} E$ are continuous functions of z in the domain of analyticity $\mathscr{D}$ of $E(z)$. Using again (3.6), we see that neither the imaginary part nor the real part of E can vanish in the domain $S(M/2; 0, \pi/2) \cap \mathscr{D}$. Therefore in this domain $\operatorname{Im} E$ remains positive and also $\operatorname{Re} E$. In the same way, we have, by considering the image by the transformation $z \to z/2$ of the strip $S(M; \pi, 2\pi)$ that neither $\operatorname{Im} E$ nor $\operatorname{Re} E$ can vanish in $S(M/2; \pi/2, \pi) \cap \mathscr{D}$. Therefore $\operatorname{Im} E$ remains positive in this domain, while $\operatorname{Re} E$

remains negative. This procedure can be iterated and we get for $z \in \mathscr{D}$,

$$\operatorname{Im} E(z) > 0 \quad \text{for} \quad 0 < \operatorname{Im} z < \pi \tag{3.8}$$

$$\operatorname{Re} E(z) > 0 \quad \text{for} \quad 0 < \operatorname{Re} z < \pi/2 \tag{3.9}$$

$$\operatorname{Re} E(z) < 0 \quad \text{for} \quad \pi/2 < \operatorname{Re} z < \pi \tag{3.10}$$

The domain $\mathscr{D}$ will be described in the next section. In the previous formulas, there is no equal sign.

3.2. Analyticity Properties

We now consider

$$E(z/2) = \left[E(z) + \lambda\right]^{1/2} \tag{3.11}$$

the determination of the square root being chosen by continuity starting from the positive real axis, where $E(z) > 0$. Then (3.11) shows that since $E(z)$ is analytic for $\operatorname{Re} z > z_0$, it will stay analytic in $\operatorname{Re} z > z_0/2$, provided there are no solutions z_c of

$$E(z_c) = -\lambda \tag{3.12}$$

in the region $\operatorname{Re} z > z_0$. For λ real and positive (3.12) implies that $z_c = i\pi + \Lambda$ where Λ satisfies

$$E(\Lambda) = \lambda \tag{3.13}$$

From the previous inequalities (3.8), Λ has to be real and positive (mod $2i\pi$). From the monotonicity properties of $E(z)$ for z real and positive, we have $E(z) \geqslant a(\lambda)$, $\forall z \geqslant 0$. Therefore (3.13) has a solution, if and only if

$$\lambda \geqslant a(\lambda) \tag{3.14}$$

and the solution is unique if it exists. (3.14) has a solution when $\lambda \geqslant 2$ (see 2.11), and no solution for $0 \leqslant \lambda < 2$. A similar argument shows that there is no solution for (3.12) when $-1/4 \leqslant \lambda \leqslant 0$. As a first conclusion, we see that for $-1/4 \leqslant \lambda \leqslant 2$, $E(z)$ is analytic in $\operatorname{Re} z \geqslant 0$, and therefore $E(z)$ maps the imaginary z axis on the Julia set. At $\lambda = 2$ a bifurcation occurs, $E(z)$ has a zero at $z = (i\pi/2) \pmod{2i\pi}$, and for $\lambda > 2$, $E(z)$ is singular with a square root branch point at $z_s = \Lambda/2 + i\pi/2$ where Λ is given by

$$\Lambda = G(\lambda) \tag{3.15}$$

The function $\lambda + E(z)$ has a simple zero at $z = \Lambda + i\pi$ because

$$E'(\Lambda) > 0 \tag{3.16}$$

$E(z)$ being a monotonically increasing function of z on the positive real axis. From the functional equation, we see that $E(z)$ will have square root branch points at all points deduced through the functional Eq. (3.1) from $z_c = \Lambda + i\pi \bmod(2i\pi)$, that is,

$$z_s^{n,k} = \frac{\Lambda}{2^n} + i\frac{(2k+1)\pi}{2^n}, \qquad n = 1, 2, \ldots, \qquad k \in Z \tag{3.17}$$

The analyticity and positivity properties of $E(z)$ for $\lambda > 2$ are best visualized on Fig. 1.

The occurrence of this dense forest of "spikes" is better understood if we go back to the Böttcher function.[4] For $\lambda > 2$, this function maps the Julia set (Fatou's dust in this case) on the unit circle. However, the inverse map is analytic on the complement of the unit disk to which have been added "spikes" along rational dyadic angles, of the form $(2k+1)\pi/2^n$, the length of the "spikes" being $e^{\Lambda/2^n}$. These "spikes" are just the images of the open intervals between points of the Cantor set.

This analysis does not extend to the case $\lambda < -1/4$ where the fixed points become complex. To end this section, we give for $\lambda = 0$ and $\lambda = 2$

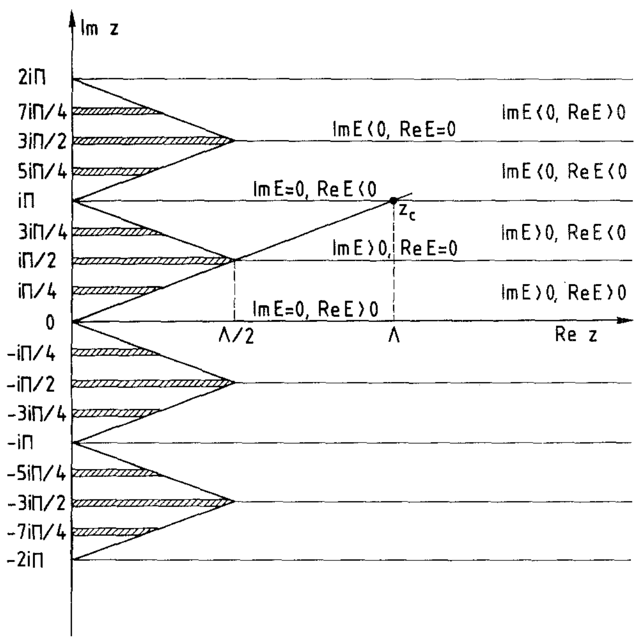

Fig. 1. Analyticity and positivity properties of the inverse Green's function for $\lambda > 2$. Only singularities for $n = 1, 2$ are displayed.

the explicit expressions of the various functions we have introduced:

$$
\begin{aligned}
\lambda = 2: \qquad & B(z) = \frac{z + (z^2-4)^{1/2}}{2}, \qquad B^{-1}(z) = z + \frac{1}{z} \\
& G(z) = \ln \frac{z + (z^2-4)^{1/2}}{2}, \qquad E(z) = 2\cosh z \\
& g(z) = \frac{1}{(1-4z^2)^{1/2}} \\
& d\mu(x) = \frac{1}{\pi} \frac{dx}{(4-x^2)^{1/2}}, \qquad x \in J = [-2, +2]
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
\lambda = 0: \qquad & B(z) = z, \qquad B^{-1}(z) = z \\
& G(z) = \ln z, \qquad E(z) = e^z \\
& g(z) = 1 \\
& d\mu(\theta) = \frac{1}{2\pi}\, d\theta, \qquad J = \{z = e^{i\theta}\}, \qquad 0 \leqslant \theta < 2\pi
\end{aligned}
\tag{3.19}
$$

4. HEURISTIC ANALYSIS OF THE BEHAVIOR OF THE GREEN'S FUNCTION NEAR THE FIXED POINT a

In this section we will sketch some formal properties of the functional equation (2.6), in order to convince the reader of the interest of introducing Mellin transforms. Setting

$$z = a + \xi \tag{4.1}$$

and

$$G(a+\xi) = \overline{G}(\xi) \tag{4.2}$$

we rewrite (2.6) as

$$\overline{G}(\xi) = \frac{1}{2}\,\overline{G}\left[2a\xi\left(1 + \frac{\xi}{2a}\right)\right] \tag{4.3}$$

We seek a formal solution of the form:

$$\overline{G}_0(\xi) = \xi^\alpha \sum_{n=0}^{\infty} \bar{g}_n(\xi/2a)^n \tag{4.4}$$

Then we have

$$\sum_{n=0}^{\infty} \bar{g}_n(\xi/2a)^n = \frac{(2a)^\alpha}{2} \sum_{k=0}^{\infty} \bar{g}_k \xi^k [1 + \xi/2a]^{k+\alpha} \tag{4.5}$$

which determines α:

$$(2a)^{\alpha} = 2$$
$$\alpha = \alpha_m = \frac{\ln 2 + 2i\pi m}{\ln 2a}, \qquad m \in Z \tag{4.6}$$

We shall consider the special solution corresponding to $m = 0$; the other solutions will be considered later. For $\lambda > -1/4$, we have

$$2a > 1 \tag{4.7}$$

and therefore

$$\alpha_0 = \frac{\ln 2}{\ln 2a} > 0 \tag{4.8}$$

From (4.5) we get

$$\bar{g}_n\left[1 - (2a)^n\right] = \sum_{k=0}^{n-1} \bar{g}_k (2a)^k \binom{k+\alpha}{n-k} \tag{4.9}$$

Fixing $\bar{g}_0 = 1$, to normalize this solution for (4.3), we see that because $2a > 1$, for $\lambda > -1/4$, all $\bar{g}_n$ can be computed from the above recursion relation. This defines a formal solution of (4.3). In fact one can prove that the series expansion (4.4) has a finite radius of convergence, but the proof is not a direct one and we do not intend to give it here, since we use here only formal arguments.

Considering now the equation (2.6):

$$G(z) = \tfrac{1}{2} G(z^2 - \lambda) \tag{4.10}$$

let $p(x)$ be any periodic function of x with period $\ln 2$:

$$p(x + \ln 2) = p(x) \tag{4.11}$$

Then one checks immediately that we get another solution of (4.10):

$$G(z) = G_0(z)\, p\left[\ln G_0(z)\right] \tag{4.12}$$

where $G_0(z)$ is any particular solution of (4.10), and we can choose for it our previous solution (4.4). If we restrict ourselves to the class of periodic functions p, which admit a Fourier expansion:

$$p(x) = \sum_{m=-\infty}^{m=+\infty} p_m e^{(2i\pi m/\ln 2)x} \tag{4.13}$$

then we have:

$$G(z) = \sum_{m=-\infty}^{m=+\infty} p_m \left[G_0(z)\right]^{1 + 2i\pi m/\ln 2} \tag{4.14}$$

In particular we see that the solution (4.6) which corresponds to $m \in Z$ is simply related to G_0 by

$$G_m(z) = \left[G_0(z) \right]^{1+2i\pi m/\ln 2} \tag{4.15}$$

Therefore, another way to write (4.14) is

$$G(z) = \sum_{m=-\infty}^{m=+\infty} p_m G_m(z) \tag{4.16}$$

which expresses the fact that the solution of (4.10) is formally expanded in the basis of the "elementary" solutions G_m. Finally, expanding the Green's function $G^{(\lambda)}(z)$ around the point $z = a$, we have using (4.14), (4.4) and (4.6):

$$\begin{aligned} \overline{G}^{(\lambda)}(\xi) &= \sum_{m=-\infty}^{m=+\infty} p_m \xi^{\ln 2/\ln 2a + 2i\pi m/\ln 2a} \left[\sum_{n=0}^{\infty} \bar{g}_n \left(\frac{\xi}{2a} \right)^n \right]^{1+2i\pi m/\ln 2} \\ &= \sum_{m=-\infty}^{+\infty} \sum_{n=0}^{\infty} p_m \bar{g}_n^m \xi^{\ln 2/\ln 2a + n + 2i\pi m/\ln 2a} \end{aligned} \tag{4.17}$$

the $\bar{g}_n^m$ being the coefficient of the expansion of the $\overline{G}_m(\xi)$. We shall consider (4.17) as an asymptotic expansion of $\overline{G}^{(\lambda)}(\xi)$ near $\xi = 0$, that is near $z = a$. Using the fact that $\overline{G}^{(\lambda)}(\xi)$ is real for $\xi > 0$ and that

$$G_m^*(z) = G_{-m}(z) \tag{4.18}$$

which implies

$$\bar{g}_n^{-m} = \bar{g}_n^{*m} \tag{4.19}$$

we see that

$$p_{-m} = p_m^* \tag{4.20}$$

Setting

$$p_m \bar{g}_n^m = \rho_m^n e^{i\varphi_m^n} \tag{4.21}$$

where ρ_m^n and φ_m^n are real numbers and

$$\varphi_0^n = 0 \tag{4.22}$$

we get,

$$\overline{G}^{(\lambda)}(\xi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho_m^n \xi^{\ln 2/\ln 2a + n} \cos\left[\frac{2\pi m}{\ln 2} \ln \xi + \varphi_m^n \right] \tag{4.23}$$

From (4.22) or (4.17) we can also obtain an asymptotic formula for the measure "density" $\sigma(x)$. Using

$$G^{(\lambda)}(z) = \int_J \ln(z - x)\, d\mu(x) = \int_J \ln(z - x) \sigma(x)\, dx \tag{4.24}$$

or

$$\frac{dG^{(\lambda)}(z)}{dz} = G'^{(\lambda)}(z) = \int_J \frac{\sigma(x)\,dx}{z - x} \tag{4.25}$$

we see that

$$\sigma(x) = -\frac{1}{\pi} \lim_{\epsilon \to 0+} G'^{(\lambda)}(x + i\epsilon) \tag{4.26}$$

therefore $\bar{\sigma}(\xi)$ will have for asymptotic expansion an expression analogous to (4.17) except for the change of n into $n - 1$:

$$\bar{\sigma}(\xi) \sim \sum_{m=-\infty}^{m=+\infty} \sum_{n=0}^{\infty} \sigma_{m,n} \xi^{\ln 2/\ln 2a + n + 2i\pi m/\ln 2a - 1} \tag{4.27}$$

with

$$\sigma^*_{m,n} = \sigma_{-m,n} \tag{4.28}$$

Such an expression is made more transparent by introducing the Mellin transform $M(s)$ of the measure $d\mu^{(8)}$:

$$M(s) = \int_J (a - x)^s \, d\mu(x) = \int_J (a - x)^s \sigma(x)\,dx \tag{4.29}$$

Equation (4.27) shows that $M(s)$ is a meromorphic function of s:

$$M(s) = \sum_{m=-\infty}^{m=+\infty} \sum_{n=0}^{\infty} \frac{\sigma_{nm}}{s + \ln 2/\ln 2a + n + 2i\pi m/\ln 2a} + \text{regular part} \tag{4.30}$$

Therefore we expect the Mellin transform of the measure $d\mu(x)$ to be a meromorphic function with poles on a periodic semi-infinite lattice in the complex plane, defined by

$$\begin{gathered} s_{m,n} = -\alpha_0 - n + im\tau \\ n \in N, \qquad m \in Z \end{gathered} \tag{4.31}$$

where

$$\tau = \frac{2\pi}{\ln 2a}, \qquad \alpha_0 = \frac{\ln 2}{\ln 2a} \tag{4.32}$$

Conversely, if we can prove that $M(s)$ is a meromorphic function with poles at points (4.31), this result will suggest that $\sigma(x)$ will have in some sense an asymptotic expansion (4.27). This expansion is certainly only asymptotic, $\sigma(x)$ being nonzero only on a Cantor set in the case $\lambda > 2$.

5. MELLIN TRANSFORM ASSOCIATED TO A JULIA SET

5.1. Definitions

One way to give a rigorous support to the previous section, is to introduce the Mellin transform of the invariant equilibrium measure $d\mu$ associated to a Julia set J, namely,

$$M_z(s) = \int_J e^{s \ln(z-x)}\, d\mu(x) \tag{5.1.1}$$

where J is the Julia set, and z is a particular point of J. To define the $\ln(z - x)$, we cut the complex plane in the variable x by a line joining z to ∞, as in Fig. 2. We introduce the Green's function $G(x)$ for polynomial mappings, with a suitable choice of the branch of the logarithm:

$$G(x) = \int_J \ln(x - y)\, d\mu(y) \tag{5.1.2}$$

If we assume the cut in Fig. 2 to be chosen along a line which connects z to infinity without intersecting J, we can rewrite (5.1.1) as

$$M_z(s) = \frac{1}{2i\pi} \oint_\Gamma e^{s \ln(z-x)}\, dG(x) \tag{5.1.3}$$

where Γ is a contour which encircles the Julia set and passes through z, as shown in Fig. 2. Clearly all formulas (5.1.1) or (5.1.3) define $M_z(s)$ as an analytic function of s as shall be seen in the sequel.

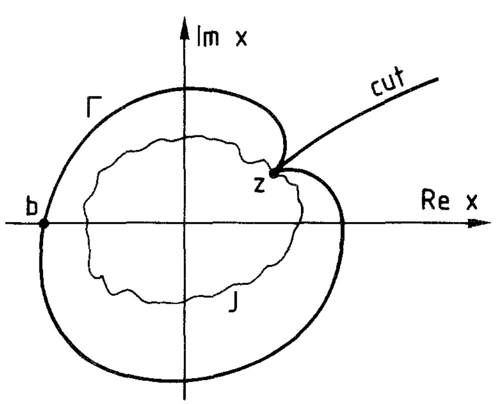

Fig. 2. The Julia set and the cut of the logarithmic function in the definition of the Mellin transform.

Although many of our results extend to more general cases as discussed in Section 6, we consider here the map:

$$T(z) = z^2 - \lambda \tag{5.1.4}$$

with λ real $> -1/4$. We choose $z = a$, where a is the following fixed point of T:

$$a = \frac{1 + (1 + 4\lambda)^{1/2}}{2} \tag{5.1.5}$$

with

$$a = a^2 - \lambda \tag{5.1.6}$$

We shall therefore drop the index z and write

$$M(s) = \int_J e^{s \ln(a-x)}\, d\mu(x) \tag{5.1.7}$$

the logarithm being cut in the x complex plane from a to $+\infty$, and defined by

$$\ln(a - x) = \ln|a - x| - i\pi + i \operatorname{Arg}(x - a) \tag{5.1.8}$$

where the $\operatorname{Arg} z$ is by definition between $-\pi$ and $+\pi$. The invariance of $d\mu$ under the change $x \to -x$ allows us to write

$$M(s) = \int_J e^{s \ln(a+x)}\, d\mu(x) \tag{5.1.9}$$

where

$$\ln(a + x) = \ln|a + x| + i \operatorname{Arg}(a + x) \tag{5.1.10}$$

Before going any further, we notice that, if we consider the first preimage of a, distinct from a itself, that is $-a$, we have

$$M_{-a}(s) = \int_J e^{s \ln(-a-x)}\, d\mu(x) \tag{5.1.11}$$

where the cut is now from $-\infty$ to $-a$, and

$$\ln(a - x) = \ln|a + x| + i\pi + i \operatorname{Arg}(a + x) \tag{5.1.12}$$

Comparison of (5.1.10) and (5.1.12) shows that

$$M_{-a}(s) = e^{i\pi s} M(s) \tag{5.1.13}$$

therefore $M_{-a}(s)$ displays the same analyticity properties as $M(s)$. In the same way, taking the preimages of $-a$, that is, $\pm(\lambda - a)^{1/2}$, we have, using the invariance of the measure,

$$M_{(\lambda-a)^{1/2}}(s) = \int_J \exp\left\{ s \ln\left[(\lambda - a)^{1/2} - x \right] \right\} d\mu(x) \tag{5.1.14}$$

$$= \tfrac{1}{2} \int_J \exp\left\{ s \ln\left[(\lambda - a)^{1/2} - (\lambda - x)^{1/2} \right] \right\} d\mu(x)$$

$$+ \tfrac{1}{2} \int \exp\left\{ s \ln\left[(\lambda - a)^{1/2} + (\lambda - x)^{1/2} \right] \right\} d\mu(x) \tag{5.1.15}$$

$$= \tfrac{1}{2} \int_J \exp\left[s \ln(x - a) \right]$$

$$\times \exp\left\{ -s \ln\left[(\lambda - a)^{1/2} + (\lambda - x)^{1/2} \right] \right\} d\mu(x)$$

$$+ \tfrac{1}{2} \int_J \exp\left\{ s \ln\left[(\lambda - a)^{1/2} + (\lambda - x)^{1/2} \right] \right\} d\mu(x) \tag{5.1.16}$$

In the latter equation, the second integral defines an entire function of s, except for $\lambda = 2$, where it is equal to the first up to a factor $e^{i\pi s}$. The analyticity properties of the first integral in (5.1.16) is the same as for $M(s)$. It is not difficult to convince oneself that, with suitable choices for the cuts, all the M functions attached to preimages of a will have the same type of singularities located at the same points. Of course one should not draw any conclusion for the closure of these preimages, which is the whole Julia set.

5.2. Meromorphic Properties of $M(s)$

Since $\int_J d\mu$ is finite, it is clear from (5.1.7) that $M(s)$ is analytic in $\operatorname{Re} s > 0$. In order to continue $M(s)$ analytically in $\operatorname{Re} s < 0$, we shall first derive a functional relation for it. In this section we will consider λ real and positive. It is convenient to split the Julia set into two parts: $J^{\pm}$, where the $+$ sign refers to the set with positive real part, and the $-$ sign to the one with negative real part. These two subsets are symmetric of one another with respect to the $\operatorname{Im} z$ axis. Starting from

$$M(s) = \int_J e^{s \ln(a-x)} d\mu(x) = \int_{J^+} e^{s \ln(a-x)} d\mu + \int_{J^-} e^{s \ln(a-x)} d\mu \tag{5.2.1}$$

we observe that

$$H(s) = \int_{J^-} e^{s \ln(a-x)} d\mu(x) \tag{5.2.2}$$

is an entire function of s, since $a \notin J^-$. Making use of the invariance of the measure, we get, for $\mathrm{Re}\, s > 0$,

$$M(s) = \int_J e^{s \ln(a^2 - x^2)}\, d\mu(x) = 2\int_{J^+} e^{s \ln(a^2-x^2)}\, d\mu(x)$$
$$= 2(2a)^s \int_{J^+} e^{s \ln(a-x)} \left[1 - \frac{(a-x)}{2a} \right]^s d\mu(x) \qquad (5.2.3)$$

Now for $\lambda > 0$, $x \in J^+$, we have $(2a) > 1$ and

$$\left| \frac{a-x}{2a} \right| \leqslant \left| \frac{a - (\lambda - a)^{1/2}}{2a} \right| < 1 \qquad (5.2.4)$$

which allows us to expand $\{1 - [(a-x)/2a]\}^s$ in an absolutely and uniformly convergent series. Therefore we get for $\mathrm{Re}\, s > 0$

$$M(s) = 2(2a)^s \sum_{j=0}^{\infty} \binom{s}{j} \frac{(-1)^j}{(2a)^j} \left[M(s+j) - H(s+j) \right] \qquad (5.2.5)$$

which can be rewritten as

$$\left[1 - 2(2a)^s\right] M(s) = -2(2a)^s H(s) + 2(2a)^s$$
$$\times \sum_{j=1}^{\infty} \binom{s}{j} \frac{(-1)^j}{(2a)^j} \left[M(s+j) - H(s+j) \right] \qquad (5.2.6)$$

The right-hand side is easily shown to be convergent for $\mathrm{Re}\, s > -1$. $H(s)$ being entire in s, $M(s)$ being analytic in $\mathrm{Re}\, s > 0$, and $\binom{s}{j}$ being a polynomial in s, we see that (5.2.6) allows one to continue $M(s)$ analytically from $\mathrm{Re}\, s > 0$ down to $\mathrm{Re}\, s > -1$ as a meromorphic function, with poles at

$$s_{0,m} = s_0 + im\tau, \qquad m \in Z \qquad (5.2.7)$$

where

$$s_0 = -\frac{\ln 2}{\ln 2a}$$
$$\tau = \frac{2\pi}{\ln 2a} \qquad (5.2.8)$$

Repeating the argument, we find that $M(s)$ extends into a meromorphic function in $\mathrm{Re}\, s > -(n+1)$, with poles at

$$s_{n,m} = s_0 - n + im\tau, \qquad n \in N, \quad m \in Z \qquad (5.2.9)$$

Since this is true for any $n > 0$, we conclude that $M(s)$ is meromorphic in the complex plane $\mathbb{C}$, with poles on a semi-infinite rectangular lattice $\mathscr{L}$.

For $\lambda > -1/4$, we introduce the entire function:

$$\pi(s) = \prod_{n=0}^{\infty} \left[1 - \tfrac{1}{2}(2a)^{-s-n} \right] \qquad (5.2.10)$$

The infinite product converges since $2a > 1$. We see that the zeros of this function are all simple and located precisely on the semi-infinite lattice $\mathscr{L}$. When $\lambda > 0$, we have seen that $M(s)$ has only simple poles on $\mathscr{L}$. Therefore we can express

$$M(s) = \frac{\epsilon(s)}{\pi(s)} \tag{5.2.11}$$

where $\epsilon(s)$ is an entire function. We shall see later how to extend these results to more general values of λ.

5.3. Integral Representation for $M(s)$, Residue Properties

We start from formula (5.1.3):

$$M(s) = \int_J e^{s\ln(a-x)}\, d\mu(x) = \frac{1}{2i\pi}\oint_\Gamma e^{s\ln(a-x)}\, dG(x) \tag{5.3.1}$$

Going to the inverse function $E(y)$ of the Green's function:

$$\begin{aligned} G(x) &= y \\ x &= E(y) \end{aligned} \tag{5.3.2}$$

we get for $\operatorname{Re} s > 0$

$$M(s) = \frac{1}{2i\pi}\int_{\bar\Gamma} e^{s\ln[a-E(y)]}\, dy \tag{5.3.3}$$

the integral being taken along a path $\bar\Gamma$ shown in Fig. 3.

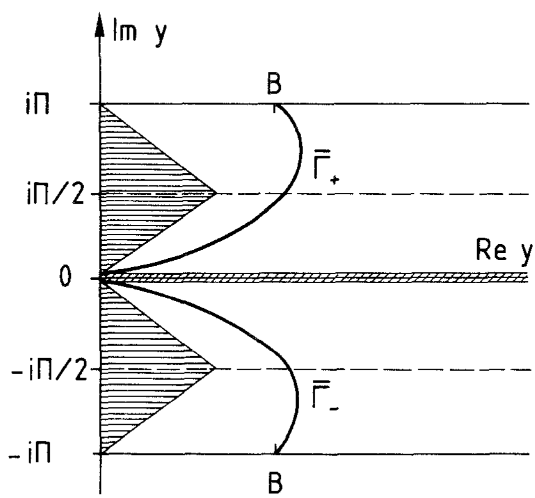

Fig. 3. The path $\bar\Gamma$ in the complex plane. B is the image (periodic) of b in Fig. 2.

Rewriting (5.3.3) for $\mathrm{Re}\, s > 0$, we get

$$M(s) = \frac{1}{2i\pi} \int_{\Gamma_+} \{ e^{s \ln[a - E(y)]} - e^{s(-i\pi + y)} \} \, dy$$
$$+ \frac{1}{2i\pi} \int_{\bar{\Gamma}} (e^{s \ln[a - E(y)]} - e^{s(i\pi + y)}) \, dy + \frac{\sin \pi s}{\pi s} \tag{5.3.4}$$

We can now push B toward infinity (see Fig. 3) along the line $\mathrm{Im}\, y = \pm \pi$, and we get

$$M(s) = -\frac{\sin \pi s}{\pi} \int_0^\infty \left[e^{s \ln[E(y) - a]} - e^{sy} \right] dy + \frac{\sin \pi s}{\pi s} \tag{5.3.5}$$

This representation is valid for $0 < \mathrm{Re}\, s < 1$, as can be seen using the expansion (2.16). Therefore both sides in (5.3.5) represent the same analytic function.

Now we cut the integral in the right-hand side of (5.3.5) into two parts, one integrated over $(0, A)$, the other over (A, ∞). Defining

$$M(A, s) = \int_0^A \left[E(y) - a \right]^s \tag{5.3.6}$$

one finds

$$M(s) = -\frac{\sin \pi s}{\pi} M(A, s) + \frac{\sin \pi s e^{As}}{\pi s}$$
$$- \frac{\sin \pi s}{\pi} \int_A^\infty \left\{ \left[E(y) - a \right]^s - e^{sy} \right\} dy \tag{5.3.7}$$

which shows that the poles and residues of $M(s)$ and $(-\sin \pi s / \pi) M(A, s)$ coincide.

Using the functional relation (2.15), we can write for $\mathrm{Re}\, s > 0$:

$$M(Y/2^n, s) = \int_0^{Y/2^n} \left[E(x) - a \right]^s ds$$
$$= 2(2a)^s \int_0^{Y/2^{n+1}} \left[E(x) - a \right]^s \left[1 + \frac{E(x) - a}{2a} \right]^s dx \tag{5.3.8}$$

Since $E(x)$ is an increasing function of x, we see that, if we choose n large enough so that

$$E(Y/2^{n+1}) < 3a \tag{5.3.9}$$

we can expand the integrand of (5.3.8) in a uniformly and absolutely convergent series. Thus we get

$$M(Y/2^n, s) = 2(2a)^s \sum_{j=0}^\infty \binom{s}{j} \frac{1}{(2a)^j} M\left(\frac{Y}{2^{n+1}}, s + j \right) \tag{5.3.10}$$

Defining

$$W(\alpha,s) = \int_{\alpha/2}^{\alpha} \left[E(x) - a \right]^s dx \tag{5.3.11}$$

we have

$$M(Y/2^n,s) = M(Y/2^{n+1},s) + W(Y/2^n,s) \tag{5.3.12}$$

Therefore

$$\begin{aligned} &\left[1 - 2(2a)^s\right] M(Y/2^{n+1},s) \\ &\quad = -W(Y/2^n,s) + 2(2a)^s \sum_{j=1}^{\infty} \binom{s}{j} \frac{1}{(2a)^j} M\left(\frac{Y}{2^{n+1}}, s+j\right) \end{aligned} \tag{5.3.13}$$

The right-hand side is analytic for $\operatorname{Re} s > -1$; therefore, the left-hand side is also analytic and this formula can be used to evaluate the residues. Moreover, each term in the series goes to zero when n goes to infinity, and since the series is uniformly convergent, only the first term on the right-hand side remains nonvanishing when n goes to infinity. Therefore we get the following representation for the residues. The poles at

$$s = s_{0,m} = s_0 + i\tau m \tag{5.3.14}$$

have for residues

$$r = r_{0,m} = -\frac{\sin(\pi s_{0,m})}{\pi \ln 2a} \lim_{n\to\infty} \int_{Y/2^{n+1}}^{Y/2^n} \left[E(x) - a \right]^{s_{0,m}} dx \tag{5.3.15}$$

Using in (5.3.13) the same kind of argument given earlier, to obtain the position of the poles (5.2.8) for $M(s)$, we see that $M_A(s)$ has its poles at the same points. In particular when s_0 is not an integer, which implies $\lambda \neq 0$, there are no poles at negative integers. As a consequence (5.3.7) shows that $M(s)$ vanishes at negative integers. This result, usually called "trace identities",[8] will be generalized to more general polynomial maps in the next chapter. Notice here that, from the above arguments, meromorphy properties of $M(s)$ persist for all $\lambda > -1/4$. The restriction $\lambda > 0$ in the previous section was only useful to simplify bounds like (5.2.4).

Let us end this section with some properties of the residues, which are established in an Appendix.

(i) The residues of the poles of $M(s)$ on the real negative axis, are strictly positive for $\lambda > 0$. In particular they do not vanish, which is not obvious from (5.3.15).

(ii) In formula (5.3.15), the right-hand side is independent of Y, since the integrand can be shown to be related to a periodic function of Y, with period $\ln 2$.

In the particular cases $\lambda = 0$, $\lambda = 2$, we have, respectively,

$$M(s) = 1 \tag{5.3.16}$$

$$M(s) = 4^s \frac{\Gamma(s + 1/2)}{\Gamma(s + 1)} \tag{5.3.17}$$

When $\lambda = 2$, $4^{-s}M(s)$ is a Stieltjes meromorphic function with poles at $s_{n,0} = -(n + 1/2)$, $n = 0, 1, 2, \ldots$ and residues:

$$r_{n,0} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n + 1/2)}{\Gamma(n + 1)} \sim \frac{1}{\pi\sqrt{n}}, \qquad n \text{ large} \tag{5.3.18}$$

One checks immediately that they are positive. In the case $\lambda = 2$, there are no complex poles, which means that the residues vanish except on the real axis.

6. GENERALIZATION TO RATIONAL TRANSFORMATION

For convenience, we consider first rational transformations with an attractive fixed point at infinity, and a real repulsive fixed point z_0 with real and positive multiplicator. Furthermore we assume that it is possible to join z_0 to ∞ along the real axis without encountering the Julia set J. We shall comment on these restrictions at the end of this section.

Let μ be the unique invariant balanced probability measure[3] associated with J, and define now the Mellin transform relative to the point z_0 by

$$M_{z_0}(s) = \int_J e^{s \ln(z_0 - x)}\, d\mu(x) \tag{6.1}$$

Here we take the cut of the logarithm to be along the real axis from z_0 to ∞. Let now be $T_i^{-1}(z)$, $i = 1, 2, \ldots, N$ be the N inverse branches of $T(z)$. In particular we define $T_1^{-1}(z)$ to be the special inverse branch for which

$$T_1^{-1}(z_0) = z_0 \tag{6.2}$$

We notice that this branch is unique, since otherwise z_0 would be superstable, violating the hypothesis that z_0 is a repulsive fixed point. Consider now ΔJ as the subset of J contained in a neighborhood K of z_0, where K is chosen small enough to insure that $T^{-1}(K) \subset K$, and *a fortiori* $T^{-1}(\Delta J) \subset \Delta J$, which is possible since z_0 is repulsive. We have

$$M_{z_0}(s) = \int_{\Delta J} e^{s \ln(z_0 - x)}\, d\mu(x) + \int_{C_{\Delta J}} e^{s \ln(z_0 - x)}\, d\mu(x) \tag{6.3}$$

Under the previous hypothesis, from which follows in particular the fact that the Julia set is compact, the second integral defines an entire function of s. Setting now

$$\Delta M(s) = \int_{\Delta J} e^{s \ln(z_0 - x)} \, d\mu(x) \tag{6.4}$$

it is clear that all singularities of $M(s)$ in the open s plane are contained in $\Delta M(s)$. The balance and invariance properties of μ allow one to write

$$\mu(E) = \mu\left[T^{-1}(E)\right] = N\mu\left[T_1^{-1}(E)\right] \tag{6.5}$$

and

$$\int_E f(x) \, d\mu = \int_{\underset{i}{U} T_i^{-1}(E)} f(T(x)) \, d\mu = N \int_{T_1^{-1}(E)} f(T(x)) \, d\mu \tag{6.6}$$

for any measurable set E and $f \in L_J^1(d\mu)$. Thus

$$\Delta M(s) = N \int_{T_1^{-1}(\Delta J)} e^{s \ln[z_0 - T(x)]} \, d\mu(x)$$

$$= N \int_{T_1^{-1}(\Delta J)} e^{s[\ln(z_0 - x)]} \left[\frac{T(z_0) - T(x)}{z_0 - x} \right]^s d\mu(x) \tag{6.7}$$

where we have also chosen ΔJ small enough so that the rational fraction $\overline{T}_{z_0}(x) = [T(z_0) - T(x)]/(z_0 - x)$ has no zeros or poles in the neighborhood K of ΔJ. This implies that $[\overline{T}_{z_0}(x)]^s$ is an analytic function of x in K. Then

$$\left[T_{z_0}(x)\right]^s = \left[T'(z_0)\right]^s \left[1 + \sum_{j=1}^{n-1} t_j (x - z_0)^j + (x - z_0)^n R_n(x, s) \right] \tag{6.8}$$

where $R_n(x, z_0)$ is a uniformly bounded function for x in K and for s in any compact subset of the s plane. Consequently,

$$\Delta M(s) = N\left[T'(z_0)\right]^s \left[\sum_{j=1}^{n-1} t_j \Delta M_1(s + j) + \int_{T_1^{-1}(\Delta J)} (z_0 - x)^{s+n} R_n(x, s) \, d\mu \right] \tag{6.9}$$

where

$$\Delta M_1(s) = \int_{T_1^{-1}(\Delta J)} e^{s \ln(z_0 - x)} \, d\mu \tag{6.10}$$

and $t_0 = 1$. We notice that the last integral in (6.9) is analytic for $\mathrm{Re}\, s > 0$. Writing (6.4) as

$$\Delta M(s) = \Delta M_1(s) + \Delta M_2(s) \tag{6.11}$$

where

$$\Delta M_2(s) = \int_{C_{T^{-1}(\Delta J)}} (z_0 - x)^s \, d\mu \tag{6.12}$$

it is clear that the above function is entire in s. Substituting (6.11) into (6.9) yields

$$\begin{aligned}[1 - NT'(z_0)^s]M_1(s) \\ = N[T'(z_0)]^s \\ \times\left[\sum_{j=1}^{n-1} t_j \Delta M_1(s+j) + \int_{T^{-1}(\Delta J)} (z_0 - x)^{s+n} R_n(x,s)\, d\mu(x) - \Delta M_2(s)\right]\end{aligned} \tag{6.13}$$

Since the right-hand side of (6.13) is analytic for $\operatorname{Re} s > -1$, this implies that $M_1(s)$ and therefore $M(s)$ will be meromorphic for $\operatorname{Re} s > -1$ with poles at $s_{0,m}$:

$$s_{0,m} = \frac{-\ln N + 2i\pi m}{\ln T'(z_0)}, \qquad m \in Z \tag{6.14}$$

Repeating this argument we see that $M(s)$ is a meromorphic function with poles at

$$s_{p,m} = s_{0,m} - p, \qquad p \in N \tag{6.15}$$

Let us now comment on our hypotheses.

(i) If $T'(z_0)$ is real negative, we can apply the previous argument not to T but to the second iterate $T(T(z))$, for which the multiplicator is real and positive, and which has the same Julia set, the same fixed point z_0, and the same invariant measure, as $T(z)$ itself.

(ii) When z_0 belongs to a cycle of order p, one applies the argument on the pth iterates.

(iii) More generally, for a complex fixed point of a rational complex transformation, one can still apply the argument on a suitable iterate when the multiplicator is a root of unity times a real number > 1.

(iv) A more serious problem in the general case of a complex transformation is the question of accessibility, that is, the question of joining the fixed point on the Julia set to infinity through a line that does not encounter the Julia set J. Moreover we need to be able to deform the initial cut to the transformed cut without meeting the Julia set. In such a case our argument will apply and give a precise meaning to the equation

$$N(T'(z_0))^s = 1 \tag{6.16}$$

the solution of which will be of the form (6.14), with a suitable determination for the exponentiation.

We think that these problems are related to the local structure of the Julia set.[21] However, we think that the meromorphic property of $M(s)$ may be quite general. The reason is that we can cut the complex plane in

(6.1) along a logarithmic spiral centered around z_0 and invariant under the multiplication $z \to (T'(z_0))z$. Going from one sheet to the other, multiplies $M(z)$ by a factor $e^{2i\pi ks}$, k integer, so, unless the Julia set covers the whole plane,[22] one can think to draw such a cut starting from z_0 and going to infinity. The idea is then that, in the vicinity of z_0, the cut is itself invariant under T, and then the meromorphy would follow from an argument similar to (6.13), provided we can also obtain in this case the suitable minimal analyticity domain for $M(s)$ required to start the argument.

7. TRACE IDENTITIES FOR POLYNOMIAL TRANSFORMATIONS

We keep here the same assumptions for $T(z)$ that were given at the beginning of Section 6, and we further assume that $T(z)$ is a polynomial. z_0 is again a real repulsive fixed point with a real positive multiplier.

Using (5.13) one has

$$M_{z_0}(s) = \frac{1}{2i\pi} \oint_\Gamma e^{s \ln(z_0 - x)} G'(x)\, dx \tag{7.1}$$

where now $G'(x)$ satisfies the following functional equation:

$$G'(x) = \frac{T'(x)}{N} G'(T(x)) \tag{7.2}$$

Here N is the degree of the polynomial T. Opening the contour Γ as in Fig. 2, and taking into account that $G'(x) \sim 1/x$ at infinity, we see that for $0 < \operatorname{Re} s < 1$, we have, as in (5.3.5),

$$M_{z_0}(s) = -\frac{\sin \pi s}{\pi} \int_{z_0}^{\infty} \left[(x - z_0)^s - e^{sG(x)} \right] G'(x)\, dx + \frac{\sin \pi s}{\pi s} \tag{7.3}$$

Dropping the index z_0, we write

$$M(s) = -\frac{\sin \pi s}{\pi} \left[M_1(s) + M_2(s) \right] + \frac{\sin \pi s}{\pi s} \tag{7.4}$$

where

$$M_1(s) = \int_{z_0}^{y} (x - z_0)^s G'(x)\, dx \tag{7.5}$$

and

$$M_2(s) = \int_y^{\infty} \left[(x - z_0)^s - e^{sG(x)} \right] G'(x)\, dx - \left(\frac{e^{sG(y)} - 1}{s} \right) \tag{7.6}$$

Noticing that $M_2(s)$ is analytic for $\operatorname{Re} s < 1$, we shall investigate the analytic structure of $M_1(s)$. Making the change of variable $x \to T(x)$ and using (7.2) yields

$$M_1(s) = N\int_{z_0}^{T_1^{-1}(y)} (Tx - z_0)^s G'(x)\,dx \tag{7.7}$$

where again T_1^{-1} is the inverse branch of T such that $T_{z_0}^{-1} = z_0$. This in turn can be written

$$M_1(s) = N\int_{z_0}^{T_1^{-1}(y)} (x - z_0)^s \left[\frac{T(x) - z_0}{x - z_0}\right]^s G'(x)\,dx \tag{7.8}$$

Now using once more the arguments given in Section 5, we find that $M_1(s)$ is a meromorphic function with poles given only by (6.14) and (6.15). Provided $T'(z_0) \neq N^{1/p}$, for any positive integer p, $M(s)$ will vanish at all negative integers:

$$M(-n) = 0, \qquad n = 1, 2, \ldots \tag{7.9}$$

Such a relation, usually called a "trace identity",[8] will give us sum rules which we plan to analyze in a later work.

8. CONCLUSION

Let us conclude this work with some comments on its mathematical and physical aspects.

8.1. Mathematical Aspects

Throughout this paper, we have considered an invariant and balanced measure μ on a Julia set J generated by the rational fraction: $T(z) = N(z)/D(z)$, of degree N. An intuitive description of this measure is obtained from the asymptotic distribution of the predecessors of an arbitrary point in the complex plane: at order k, we consider its N^k preimages, on which sit Dirac measures with weight N^{-k}, added together to build up a discrete probability measure. The invariant measure is thought to be the weak limit of this sequence.[2] When such a limit exists, we can consider the associated "free energy" $F(y)$ defined in (1.22), or more conveniently, the generating function of the moments:

$$g(y) = y\int_J \frac{d\mu(x)}{y - x} = \sum_{n=0}^{\infty} \frac{g_n}{y^n} = \sum_{n=0}^{\infty} \frac{1}{y^n}\int_J x^n\,d\mu(x) \tag{8.1}$$

which satisfies

$$g(y) = \frac{y}{N}\,\frac{T'(y)}{T(y)}\,g(T(y)) + \frac{y}{N}\,\frac{D'(y)}{D(y)} \tag{8.2}$$

This last equation permits recursive generation of the moments g_n provided $g(y)$ is analytic in the vicinity of the point at infinity, which is true when J is bounded, that is, when the point at infinity is an attractive fixed point of T. In fact the point at infinity plays no special role since a linear fractional change of coordinates allows us to extend these considerations to any T having an attractive fixed point. We must certainly exclude the case where J is the full complex plane,[22] and we leave aside the problem when only indifferent or repulsive fixed points occur. The quadratic polynomial case already exhibits a sufficiently complicated situation[23,24] to give an idea of the difficulties of the general case.

The equations (8.2) and (1.22) generalize to rational fractions the equation (1.5) valid in the polynomial case. Notice that (8.2) has the substantial advantage of being uniform in the vicinity of the point at infinity. The existence of the weak limit mentioned above has some relation with the convergence of the iterations of Eq. (1.22):

$$F(y)=\sum_{k=0}^{\infty}\frac{1}{N^{k+1}}\ln\left[D\left(T^k(y)\right)\right] \tag{8.3}$$

which is easily shown to be convergent in each basin of attraction of any attractive fixed point or cycle of T.

In the present work, we have mainly been interested in the behavior of the above function $F(y)$ near the boundary of its analyticity domain, a question to which a partial answer can be given using the functional equation (1.22). This problem is related to the asymptotic behavior of the entire (or meromorphic) functions introduced by Poincaré and Picard.[20] Here we have analyzed the local structure of $F(y)$ near a repulsive fixed point through the analyticity properties of the Mellin transform (6.1), which has been shown to be meromorphic. We further notice that when the Julia set is real and when z_0 is the rightmost fixed point, the Mellin transform coincides with the following function:

$$\tilde{M}_{z_0}(s)=\int_J |z_0-x|^s\,d\mu(x) \tag{8.4}$$

We may ask whether $M_{z_0}(s)$ has also meromorphic properties in the general case, and also analyze linear superpositions of such functions on the various repulsive fixed points of J. Our intuition is that such a generalized function could be connected with the ζ function approach in global analysis of dynamical systems,[25] and to their generalizations considered in the study of the Hausdorff dimension of Julia sets.[26]

We consider now the case where the Julia set is real. We shall discuss the content of the approximate expansion (4.23) of the integrated measure's density. A first consistency consequence of (4.23) is to provide an asymptotic behavior for the moments of the measure, which can be recursively computed from Eq. (1.5), through an expansion around the point at infinity. The algebra is straightforward and was already given in a different context.[27] From (4.27), we deduce that in the quadratic case, for λ real, and n large

$$g_{2n} = \int_J x^{2n}\, d\mu(x) \sim a^{2n}\left(\frac{a}{2n+1}\right)^{\alpha_0} \times\left[C_0 + \sum_{k=1}^{\infty} C_k \cos k\tau \ln\left(\frac{2n+1}{a}\right) + O\left(\frac{1}{n}\right)\right] \tag{8.5}$$

where α_0 and τ are given in (4.32) and a in (2.11). The oscillatory part is well observed in numerical calculations and the amplitude of the oscillatory part is significant (of the order of 10% of the constant part for $\lambda = 3$ and $\lambda = 1$ for instance). Of course this oscillatory part disappears when $\lambda = 2$. This gives a numerical confirmation of the consistency of the scheme we have developed.

The existence of complex poles in the Mellin transform is also confirmed through a multipoint Pade analysis of $M_a(s)$ itself. In fact the positions of the complex poles closest to the origin are well reproduced (when $\lambda \neq 2$) by the position of the poles of the Pade approximants which take the same values as $M_a(s)$ for n positive integer ($0 \leqslant n \leqslant N$). However the complexity of the analytic structure of $M_a(s)$ which arises from the lattice of poles, necessitates further work for a precise evaluation of the residues.

8.2. Physical Aspects

The physical consequences are twofold: first in quantum mechanical systems, second in statistical mechanics. We will summarize here the content of the quadratic mapping Hamiltonian model,[16] which we regard as a paradigmatic example of a Schrödinger operator with purely singular continuous spectrum. One considers a one-dimensional quantum mechanical system on a lattice (integers). Supposing the existence of a decimation operator D, which acts on the wave function ψ_n at site n as

$$D\psi_n = \psi_{2n} \tag{8.6}$$

that is, by suppressing a component out of two in the wave function, it is then possible to show the existence of an Hamiltonian H subject to the algebraic constraint

$$HD = D(H^2 - \lambda), \qquad \lambda > 2 \tag{8.7}$$

This condition is fulfilled by choosing for H a tridiagonal Jacobi matrix.[6]

Then if $|\psi\rangle$ is a (quasi) eigenstate of H with eigenvalue E, $D|\psi\rangle$ is also a (quasi) eigenstate with value $E^2 - \lambda$. The main results about this model are as follows:

(i) The spectrum is purely singular; its support reduces to a Cantor set of Lebesgue measure zero, consisting of all points of the form

$$\left(\lambda \pm \left(\lambda \pm (\lambda \pm \dots)^{1/2}\right)^{1/2}\right)^{1/2} \tag{8.8}$$

which is nothing else than the Julia set associated to

$$T(z) = z^2 - \lambda, \qquad \lambda > 2 \tag{8.9}$$

(ii) The wave functions are the set of orthogonal polynomials $P_n(E)$ with respect to the equilibrium measure defined on the Julia set.

(iii) The resolvent (in the state $|\psi\rangle$ with $\psi_n = \delta_{n,0}$)

$$\left\langle \delta_0 \middle| \frac{1}{z - H} \middle| \delta_0 \right\rangle = \int_J \frac{d\mu(E)}{z - E} \tag{8.10}$$

is the generating function of the moments of $d\mu$ and in this case $\mu(E)$ is also the integrated density of states. For physical reasons (specific heat, phonons), it is of interest to study the excitation spectrum near its end point. In general one writes for the spectral density a power law, deduced from scaling arguments,[9,10] of the following form:

$$\frac{d\mu(E)}{dE} = \sigma(E) \sim C(a - E)^{\delta}, \qquad E_{\text{end}} = a \tag{8.11}$$

where δ is the spectral dimensionality.[10] One of the unexpected results of our work is that such a formula is too simple in our case, a full set of complex values of δ must be added to the previous expression (8.10). It is likely to be the same in more general Hamiltonians with singular continuous spectral measures (almost Mathieu equation, for instance). The same observation holds for mechanical systems on fractal structure.[16,30]

In statistical mechanics, the scaling laws near the critical point have to be corrected and oscillatory contributions occur which can be interpreted as an oscillatory character of the critical amplitudes. Such pathologies are usually excluded in regular translation invariant systems,[28] but must be encountered in systems when the renormalization is exact and rational.[12] This may also be an explanation for the oscillatory behavior encountered in calculation of thermodynamical quantities by approximate renormalization group techniques.[29] One may wonder whether such oscillations should persist in disordered system[17] or in complex magnetic fields,[29] when considering the exact models and not their truncated (or hierarchical) approximations.

ACKNOWLEDGMENTS

We want to thank Drs. B. Derrida, C. Itzykson, D. Ruelle, and A. Voros for fruitful discussions.

APPENDIX: PROPERTIES OF THE RESIDUES

Starting from formula (5.3.15), we get, after a change of variable,

$$r_{0,m} = -\frac{\sin(\pi s_{0,m})}{\pi \ln(2a)} \lim_{n\to\infty} \int_y^{2y} \frac{x}{2^{n+1}} \left[E\left(\frac{x}{2^{n+1}}\right) - a \right]^{s_{0,m}} \frac{dx}{x} \tag{A1}$$

Using (2.15) and (5.1.6), one finds

$$\frac{x}{2^{n+1}} \left[E\left(\frac{x}{2^{n+1}}\right) - a \right]^{s_{0,m}} = \frac{x\left[E(x) - a\right]^{s_{0,m}}}{\prod_{i=1}^{n+1}\left\{1 + 1/2a\left[E(x/2^i) - a\right]\right\}^{s_{0,m}}} \tag{A2}$$

we define

$$F_{s_{0,m}}(x) = \lim_{n\to\infty} \frac{x}{2^{n+1}} \left[E\left(\frac{x}{2^{n+1}}\right) - a \right]^{s_{0,m}} \tag{A3}$$

It is a consequence of (A2), (2.15), (5.1.6) and the monotonicity property of $E(x)$ for x real that the above limit exists for $0 < x < \infty$. Furthermore it follows from the definition that

$$F_{s_{0,m}}(x) = F_{s_{0,m}}(2x) \tag{A4}$$

Setting $x = e^t$ and $\psi_{s_{0,m}}(x) = F_{s_{0,m}}(e^x)$ the above formula yields

$$\psi_{s_{0,m}}(t) = \psi_{s_{0,m}}(t + \ln 2) \tag{A5}$$

Thus $\psi_{s_{0,m}}$ is periodic with period $\log 2$, and we have

$$r_{0,m} = -\frac{\sin(\pi_{s_{0,m}})}{\pi \ln 2a} \int_{\ln y}^{\ln y + \ln 2} \psi_{s_{0,m}}(t)\, dt \tag{A6}$$

This equality displays the relation between the independence on y and the periodicity property of $\psi_{s_{0,m}}$.

From (A1), and provided that $\sin(\pi s_0) < 0$, which is achieved for $\lambda > 0$, we get

$$r_{0,0} > 0 \tag{A7}$$

We shall now prove that there is a simple relation between $r_{0,m}$ and $r_{n,m}$, an immediate consequence of which will be the positivity of the residues $r_{m,0}$. Using equations (5.3.6) and (5.3.11), we define

$$\gamma_n(s) = \frac{-\sin \pi s}{\pi} M\left(\frac{y}{2^{n+1}}, s\right) \tag{A8}$$

$$V_n(s) = \frac{\sin \pi s}{\pi} W\left(\frac{y}{2^n}, s\right) \tag{A9}$$

Setting

$$K(s,i) = \frac{2(2a)^s}{1-2(2a)^s} \cdot \frac{(-s)_i}{(1)_i(2a)_i} \tag{A10}$$

where the classical Pochhammer's symbol $(b)_i$ is equal to

$$(b)_i = b(b+1)\cdots(b+i-1) \tag{A11}$$

one has from (5.3.13), and for $\operatorname{Re} s > 0$

$$\gamma_n(s) = \frac{V_n(s)}{1-2(2a)^s} + \sum_{j=1}^{\infty} K(s,j)\gamma_n(s+j) \tag{A12}$$

The value of y in (A8) and (A9) has been chosen small enough, (but nonzero) to insure the absolute convergence of the series. We first observe two important properties of $K(s,i)$ defined in (A10):

$$K(s,i) \leqslant \left| \frac{2(2a)^s}{1-2(2a)^s} \right| \frac{(i)^{-\operatorname{Re} s}}{|\Gamma(-s)|(2a)^i} \left[1 + O\left(\frac{1}{i}\right) \right] \tag{A13}$$

$$K(s,i) > 0 \qquad \text{for } s \text{ real}, \quad s < s_0 \tag{A14}$$

Multiplying (A12) by $1-2(2a)^s$; then taking the limit $s \to s_{0,m} = s_0 + im\tau$, one finds

$$\lim_{n\to\infty} \lim_{s\to s_{0,m}} \left[1-2(2a)^s\right] \frac{\gamma_n(s)}{\ln(2a)} = r_{0,m} = \frac{-V_\infty(s_{0,m})}{\ln 2a} \tag{A15}$$

Now iterating (A12) once yields for $\operatorname{Re} s > 0$

$$\begin{aligned} \gamma_n(s) = {} & \frac{V_n(s)}{1-2(2a)^s} + \sum_{i=1}^{\infty} K(s,i) \frac{V_n(s+i)}{1-2(2a)^{s+i}} \\ & + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} K(s,i_1)K(s+i_1,i_2)\gamma_n(s+i_1+i_2) \end{aligned} \tag{A16}$$

If we now multiply both sides by $[1-2(2a)^s][1-2(2a)^{s+1}]$, we find, using (A13), that the above series can be continued to $\operatorname{Re} s > -2$, and that

$$\lim_{n\to\infty} \lim_{s\to s_{0,n-1}} \left[1-2(2a)^{s+1}\right] \frac{\gamma_n(s)}{\ln(2a)} = r_{1,m} = K(s_{0,m-1},1)r_{0,m} \tag{A17}$$

From (A14) we deduce for $\lambda > 0$

$$r_{1,0} > 0 \tag{A18}$$

The general case follows upon iterating (A12) n times, and multiplying by $\prod_{i=0}^{n}[1-2(2a)^{s+i}]$. The new series can be continued to $\operatorname{Re} s > s_0 - n - 1$;

thus, for $p > 0$,

$$\lim_{n\to\infty} \lim_{s\to s_{0,m-p}} \left[1 - 2(2a)^{s+p}\right] \frac{\gamma_n(s)}{\ln 2a} = r_{p,m} \tag{A19}$$

satisfies

$$\begin{aligned} r_{p,m} = r_{0,m} \Bigg\{ & K(s_{0,m-p}, p) \\ & + \sum_{i_1=1}^{p-1} K(s_{0,m-p}, i_1) K(s_{0,m-p+i_1}, p-i_1) \\ & + \sum_{i_1=1}^{p-2} \sum_{i_2=1}^{p-1-i_1} K(s_{0,m-p}, i_1) \\ & \times K(s_{0,m-p+i_1}, i_2) K(s_{0,m-p+i_1+i_2}, p-i_1-i_2) \\ & + \cdots + \\ & + \prod_{j=1}^{p} K(s_{0,m-j}, 1) \Bigg\} \end{aligned}$$

from which, using (A14), we get $r_{p,0} > 0$ for $\lambda > 0$.

REFERENCES

1. G. Julia, Mémoire sur l'iteration des fonctions rationnelles, *J. Math. Ser. 7 (Paris)* **4**:47–245 (1918); P. Fatou, Sur les équations fonctionnelles, *Bull. Soc. Math. France* **47**:161–271 (1919); **48**:33–94 (1920); **48**:208–314 (1920).
2. H. Brolin, Invariant sets under iterations of rational functions, *Ark. Mat.* **6**:103–144 (1965).
3. M. F. Barnsley, J. S. Geronimo, and A. N. Harrington, Orthogonal polynomials associated with invariant measure on Julia sets, *Bull. Am. Math. Soc.* **7**:381–384 (1982); A. Freire, A. Lopez, and R. Mañe, An invariant measure for rational maps, Maryland University preprint, 1982; M. F. Barnsley and A. N. Harrington, Moments of balanced measures on Julia set, Georgia Institute of Technology preprint, 1982.
4. M. F. Barnsley, J. S. Geronimo, and A. N. Harrington, Geometrical and electrical properties of some Julia sets, Georgia Institute of Technology preprint, 1982, submitted to *Journal of Statistical Physics*.
5. D. Bessis and P. Moussa, Orthogonality properties of iterated polynomial mappings, *Commun. Math. Phys.* **88**:503–529 (1983).
6. J. Bellissard, D. Bessis, and P. Moussa, Chaotic states of almost periodic Schrödinger operators, *Phys. Rev. Lett.* **49**:701–704 (1982).

7. D. J. Thouless, A relation between the density of states and the range of localization for one dimensional random systems, *J. Phys. C: Solid State Phys.* **5**:77–81 (1972).
8. A. Voros, The zeta function of the quartic oscillator, *Nucl. Phys.* **B165**:209–236 (1980); see also contributions by G. Parisi, A. Voros, L. Bers, D. and V. Chudnovsky in The WKB method in the Riemann Problem, Complete Integrability and Arithmetic Applications, *Lecture Notes in Mathematics*, Vol. 925 (Springer, New York, 1982).
9. D. Dhar, Lattice of effectively non integral dimensionality, *J. Math. Phys.* **18**:577–585 (1977).
10. R. Rammal and G. Toulouse, Random walks on fractal structure and percolation clusters, *J. Phys. Lett. Paris* **44**:L13–22 (1983).
11. A. N. Berker and D. Ostlund, Renormalization-group calculation of finite systems: Order parameter and specific heat for epitaxial ordering, *J. Phys. C: Solid State Phys.* **12**:4961–4975 (1979); M. Kaufman and R. B. Griffiths, Exactly soluble Ising models on hierarchical lattices, *Phys. Rev.* **B24**:496–498 (1981).
12. B. Derrida, L. De Seze, and C. Itzykson, Fractal structure of zeroes in hierarchical models, Saclay DPh.T. preprint, 1983, to appear in *J. Stat. Phys.*
13. C. N. Yang and T. D. Lee, Statistical theory of equations of state and phase transition: I. Theory of condensation, *Phys. Rev.* **87**:404–410 (1952); T. D. Lee and C. N. Yang, Statistical theory of equation of state and phase transition: II Lattice gas and Ising model, *Phys. Rev.* **87**:411–419 (1952).
14. B. Derrida, J. P. Eckmann, and A. Erzan, Renormalization group with periodic and aperiodic orbits, *J. Phys. A: Math. Gen.* **16**:893–906 (1983); B. Derrida and H. J. Hilhorst, Singular behavior of certain infinite products of 2×2 matrices, *J. Phys. A: Math. Gen.* **16**:2641–2654 (1983).
15. D. Ruelle, Thermodynamical formalism, in *Encylopaedia of Mathematics and Its Application*, Vol. 5 (Addison–Wesley, Reading, Massachusetts, 1978).
16. R. Rammal, Nature of eigenstates on fractal structures, *Phys. Rev.* **B28**:4871–4874 (1983); E. Domany, S. Alexander, D. Bensimon, and L. P. Kadanoff, Solutions to the Schrödinger equation on some fractal lattices, *Phys. Rev.* **B28**:3110–3123 (1983).
17. T. C. Lubensky, Thermal and geometrical critical phenomena in random system, in *Ill-Condensed Matter*, R. Balian et al. eds., Les Houches session XXXI, pp. 405–475 (North-Holland, Amsterdam, 1979).
18. M. F. Barnsley, J. S. Geronimo, and A. N. Harrington, On the invariant sets of a family of quadratic maps, *Commun. Math. Phys.* **88**:479–501 (1983).
19. D. Bessis, M. L. Mehta, and P. Moussa, Polynomes orthogonaux sur des ensembles de Cantor et iterations des transformations quadratiques, *C. R. Acad. Sci. (Paris)* **293**:Ser. I, 705–708 (1981); D. Bessis, M. L. Mehta, and P. Moussa, Orthogonal polynomials on a family of Cantor sets and the problem of iteration of quadratic maps, *Lett. Math. Phys.* **6**:123–140 (1982).
20. G. Valiron, *Fonction analytiques*, Chap. VII (Presses universitaires de France, Paris, 1954).
21. A. Douady, Systèmes dynamiques holomorphes, Séminaire Bourbaki No. 599, November 1982, *Asterisque* **105–106**:39–63 (1983).
22. S. Lattes, Sur l'iteration des substitutions rationnelles et les fonctions de Poincaré, *C. R. Acad. Sci. Paris* **166**:26–28 (1918).
23. B. Mandelbrot, Fractal aspects of the iterations of $z \to \lambda z(1 - z)$ for complex λ and z, *Ann. N.Y. Acad. Sci.* **357**:249–259 (1980).
24. A. Douady and J. H. Hubbard, Iteration des polynomes quadratiques complexes, *C. R. Acad. Sci. Paris* **294**:Ser. I, 123–126 (1982).
25. M. Artin and B. Mazur, On periodic points, *Ann. Math.* **81**:82–89 (1965).
26. D. Ruelle, Repellers for real analytic maps, *Ergodic Theory Dyn. Syst.* **2**:99–107 (1982).
27. M. Barnsley, D. Bessis, and P. Moussa, The diophantine moment problem and the analytic structure in the activity of the ferromagnetic Ising model, *J. Math. Phys.*

20:535–546 (1979); see equation (4.27), p. 543.

28. M. Nauenberg, Scaling representation for critical phenomena, *J. Phys. A: Math. Gen.* **8**:925–928 (1975); Th. Niemeijer and J. M. J. Van Leeuwen, Renormalization theory for Ising like spin system, in *Phase Transition and Critical Phenomena*, Vol. 6, C. Domb and M. S. Green, eds. (Academic Press, New York, 1976), pp. 425–505.
29. K. Uzelac, P. Pfeuty, and R. Julien, Yang Lee edge singularity from a real space renormalization group method, *Phys. Rev. Lett.* **43**:805–808 (1979); R. Jullien, K. Uzelac, P. Pfeuty, and P. Moussa, The Yang Lee edge singularity studied by a four level quantum renormalization group blocking method, *J. Phys. Paris* **42**:1075–1080 (1981).
30. D. Bessis, J. S. Geronimo, and P. Moussa, Complex spectral dimensionality on fractal structures, *J. Phys. Lett. Paris* **44**:L977–982 (1983).